Wavelet phase synchronization and chaoticity

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It has been shown that the so-called "wavelet phase" (or "time-scale") synchronization of chaotic signals is actually synchronization of smoothed functions with reduced chaotic fluctuations. This fact is based on the representation of the wavelet transform with the Morlet wavelet as a solution of the Cauchy problem for a simple diffusion equation with initial condition in a form of harmonic function modulated by a given signal. The topological background of the resulting effect is discussed. It is argued that the wavelet phase synchronization provides information about the synchronization of an averaged motion described by bounding tori instead of the fine-level classical chaotic phase synchronization.

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I. INTRODUCTION

Synchronization is one of general nonlinear phenomena observed in a wide range of physical, chemical, and biological processes (see, for example, the comprehensive review in the book [1]). The adjustment of self-sustained periodic oscillators can vary from phase locking to complete synchrony. The nontrivial example investigated first in [2] is the synchronization of chaotic systems, which means that the absolute value of the instantaneous phase [$\phi_1(t)$ and $\phi_2(t)$] difference of two chaotic functions or time series $f(t)_1$ and $f_2(t)$ must be a bounded function of time: $|\phi_1(t)-\phi_2(t)|<\text{const}$ for any t.

However, the definition mentioned above reveals the important problem, which is still far from a complete solution: how to introduce a phase of a chaotic oscillator. Conversely to the case of harmonic or weak-nonlinear oscillations, there are many possible approaches. The current state-of-the-art and comparison of various methods can be found in the review [3]. For systems with a simple attractor topology, one can use conventional methods: a polar angle between velocity and displacement on a phase plane or the phase, which is extracted from the result of the Hilbert [1] or wavelet [4] transforms. This case is referred as "well-defined phase."

However, if trajectories of two-dimensional attractor projection do not revolve around a unique origin, then the phase is ill defined (see the discussion in [5]).

The authors of the article [6] have mentioned that in such case, there exists a synchrony between phases $\phi(a,b)$ of the complex wavelet transform $w(a,b)=|w(a,b)|\exp[i\phi(a,b)]$, where

$$w(a,b)=\int_{-\infty}^{+\infty} f(t)e^{i\omega_0(t-b)/a}e^{-[(t-b)^2/2a^2]}\frac{dt}{\sqrt{2\pi}a}. \qquad (1)$$

Here the shift b plays the role of a time variable. The variable a is called "scale" and connected with a local period (or frequency) of the analyzed signal: for the case of simple harmonic oscillation $f(t)=\exp(i\nu_0 t)$, its wavelet transform has a line of maximum corresponding to the scale $a=\omega_0/2\pi\nu_0$.

In the paper [7], the authors argue that this detection is an example of some general phenomena named by them as "time-scale synchronization." The corresponding condition for synchronization of chaotic systems with phases $\phi_1(a,b)$ and $\phi_2(a,b)$ on a scale a_0 is $|\phi_1(a_0,b)-\phi_2(a_0,b)|<\text{const}$; i.e., the absolute value of the instantaneous phase difference must be a bounded function of time for the given scale a.

The causes for the wavelet regularization of an ill-defined phase are not revealed in the first presentations of this method [6,7] nor in the further articles (e.g., [8,9]). On the other hand, it has been shown [10] that there is no synchronization if the used in Eq. (1) central frequency ω_0 is lower than a certain critical value.

The main goal of this work is to analyze how the wavelet transform disturbs the initial signal during the continuous change in the central frequency in the sense of conservation/lack of its chaoticity as well as a topological background of the wavelet phase synchronization.

II. RESULTS

Note that the center frequency ω_0 of the standard Morlet wavelet transform [11] determines both time and scale-space resolution. In particular, it was demonstrated in [12,13] that the Morlet wavelet transform provides a better detection tool for isolated pulses and short-pulse trains when ω_0 is low and resolves better long signals (which can be represented by Fourier series) otherwise. For this reason, consider the central frequency ω_0 as an independent variable.

To facilitate interpretation, let us define the frequency variable ν with respect to the scale a by the relation $\nu=\omega_0/\pi a$. In these terms, the transform (1) can be rewritten as

$$w(\nu,\omega_0,b)=e^{i\pi\nu b}\int_{-\infty}^{+\infty} f(t)e^{-i\pi\nu t}\frac{e^{-[(t-b)^2/4\omega_0^2(1/2\pi^2\nu^2)]}}{\sqrt{4\pi\omega_0^2\dfrac{1}{2\pi^2\nu^2}}}dt. \qquad (2)$$

It has a form of the harmonic oscillation with frequency ν modulated by the time-dependent complex amplitude

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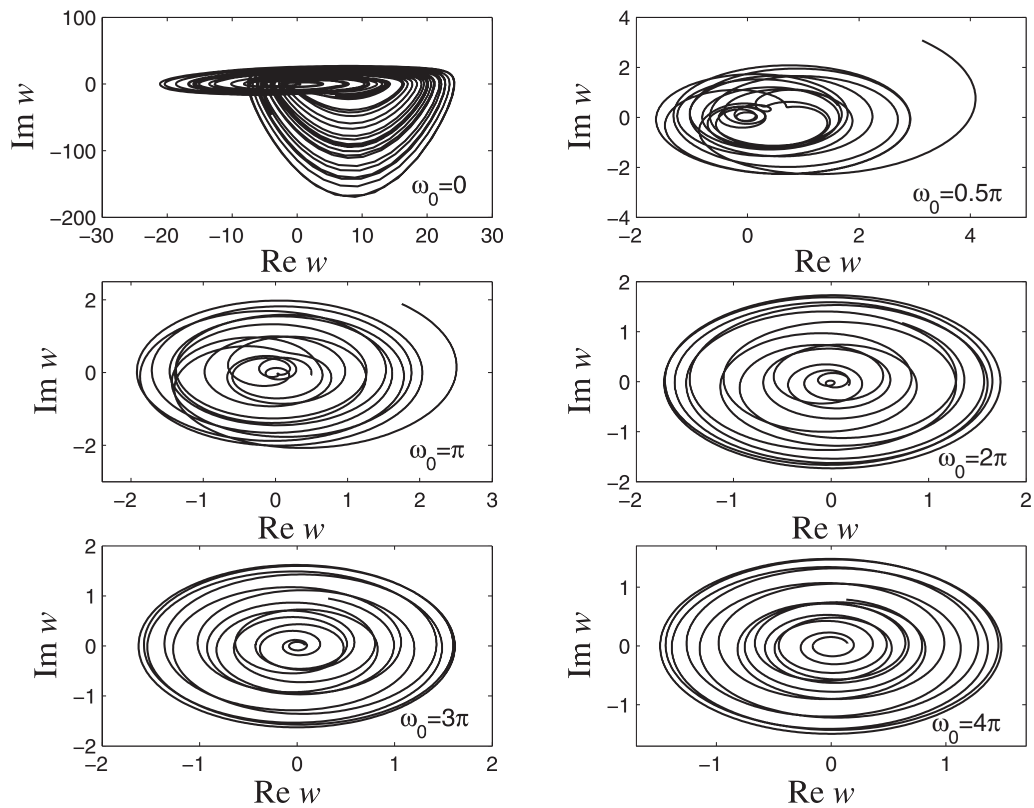

FIG. 1. Phase portraits of the Rössler oscillator dynamics in the wavelet space for various central frequencies.

$w(\nu,\omega_0,b)=u(\nu,\omega_0,b)\exp(i\pi\nu b)$. The amplitude u is the integral in Eq. (2), i.e., an integral transform with the diffusion kernel. It is well known that such a kernel tends to the Dirac delta function, if ω_0 tends to zero. Due to this property, Eq. (2) takes the form

$$w(\nu,0,b) = f(b), \tag{3}$$

if $\omega_0=0$. Note that the analyzed function $f(b)$ can be real as well as complex.

Thus, the study of synchronization in the limit $\omega_0\to 0$ does not differ from the standard conventional methods. For example, one can use the complexification in the form of a simple phase curve $f_c(t)=f(t)+i\dot{f}(t)$ or via the Hilbert transform $f_H(t)=f(t)+iH[f(t)]$. As a result, the wavelet phase in the limit $\omega_0=0$ will coincide with the usual phase of a complex number: the polar angle $\phi(t)=\arctan(\dot{f}(t)/f(t))$ or $\phi(t)=\arctan(H[f(t)]/f(t))$.

Therefore, all properties mentioned in [6–9] are induced by a finiteness of ω_0. From a mathematical point of view, let τ be a "time variable" and let b be a "space variable." Then Eq. (2) is a solution of the diffusion equation

$$\frac{\partial u}{\partial \tau} = (2\pi^2\nu^2)^{-1}\frac{\partial^2 u}{\partial b^2}, \tag{4}$$

at the "instant" $\tau=\omega_0^2$ with the diffusion coefficient $(2\pi^2\nu^2)^{-1}$. Therefore, the Morlet wavelet transform with fixed ν and variable ω_0 acts as a diffusion smoothing. Thus, the problem is close enough to the classical problem of diffusion image processing [14]. Here the transformed image is the texture defined on the plane (b,ν): $f(b)=\exp(i\pi\nu b)$.

The dependence of the diffusion coefficient on ν provides the opportunity to adjust a smoothing rate to local frequency of modulations that is a basic important property of wavelets.

From this "diffusional" point of view, it is clear that the growth of ω_0 leads to the averaging, which eliminates high-frequency fluctuations. But even these fluctuations provide a divergence of phase trajectories for slightly different initial values of a chaotic signal. The time uncertainty $\omega_0/\pi\nu$ plays a role of the corresponding quantitative measure for the averaging window. The phase shifts within this interval are indistinguishable, thus, one can detect some "synchronization."

As $\omega_0\to\infty$, the Morlet wavelet transform becomes the Fourier integral transform, with infinite time domain and zero-frequency/scale band. In this limit, two signals can be compared in terms of the presence of certain frequencies in a global sense, but neither instantaneous phase nor instantaneous frequency can be well defined; i.e., time-scale synchronization is out of question.

As an example, consider from this point of view the wavelet phase curves for the system of two coupled Rössler oscillators

$$\dot{x}_{1,2} = -\omega_{1,2}y - z_{1,2} + \epsilon(x_{2,1}-x_{1,2}),$$

$$\dot{y}_{1,2} = -\omega_{1,2}x_{1,2} + a_0 y_{1,2} - z_{1,2} + \epsilon(y_{2,1}-y_{1,2}),$$

$$\dot{z}_{1,2} = p + z_{1,2}(x_{1,2}-c), \tag{5}$$

with $\omega_1=0.98$, $\omega_2=1.03$, $a_0=0.22$, $p=0.1$, $c=8.5$, and $\epsilon=0.05$ studied in [6,7].

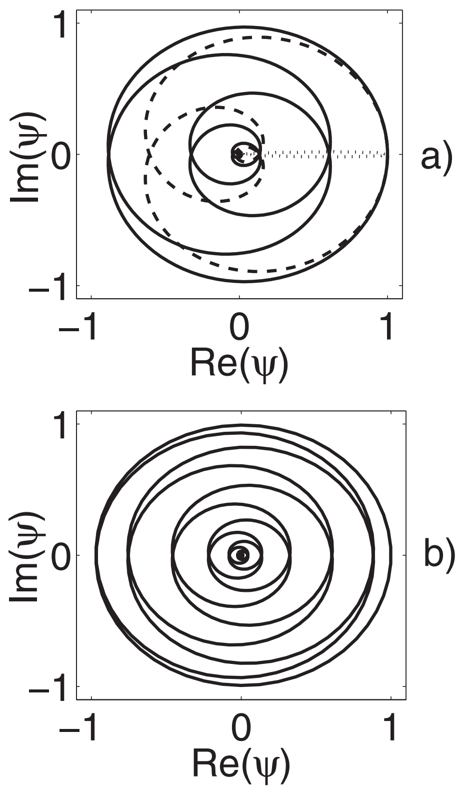

FIG. 2. The Morlet wavelet represented on a complex plane for the scale $a=1$ and various central frequencies: (a) $\omega_0=0.01$ (dotted line), $\omega_0=\pi$ (dashed line), and $\omega_0=2\pi$ (solid line); (b) $\omega_0=4\pi$.

Take the complex combination $f_{1,2}(t)=x_{1,2}(t)+\dot{x}_{1,2}$ as an initial value for Eq. (4) at the fixed inverse scale ν corresponding to the main line of maximum. Figure 1 presents wavelet phase portraits for various ω_0 in the case of scale $a=5.25$, corresponding to the time-scale synchronization described in [6,7]. The case of $\omega_0=0$ is a usual phase portrait of the Rössler oscillator in the variable $(x,\dot{x})$. With increasing ω_0, the wavelet support has no more than zero length, high harmonics are eliminated, and the phase portrait tends to more regular shape, which is almost elliptic. Here a length of radius-vector coincides with a modulus of the wavelet transform and a polar angle is a wavelet phase. Naturally, the motion along these "almost ellipses" leads to an increasingly regular change in the latter. Thus, one can detect synchronization of this "almost-regular" motion of two coupled oscillators.

III. DISCUSSION AND OUTLOOK

The considered phenomenon has also a topological explanation based on the concept of bounding tori first introduced in [15]. It has been shown there that chaotic attractors have various scales of structure: a fine set of unstable orbits at a finer level and a torus with holes, which encloses it. This bounding torus is a semipenetrable surface defining the domain from which a phase trajectory cannot escape (see the review with applications to various examples including Rössler system, in [16]). In principle, one can determine quantitative characteristics of these tori, say radii, as it has been done in [17] for the size of domain bounding Lorenz attractor. These radii could be associated with the boundaries of uncertainty window for the analyzing wavelet.

From this topological point of view, the integral (1) means that the functional product of the phase curves of transformed solution and the wavelets with given scales and time shifts. In other words, the result is nonzero in the points, where both curves have a common tangent. For example, certain phase curves for the Morlet wavelet are represented in the Fig. 2. Obviously, small values of ω_0 correspond to the very narrow ovals. In the limit $\omega_0 \to 0$, they simply tend to radius vector, which pick up a current point of the original phase curve. For larger central frequencies, the wavelet will consist of several turns on a phase plane. Therefore, it can touch an analyzed phase curve in several points, which belong to different orbits. That means that the resulting wavelet phase curve, as pictured in Fig. 1, can be considered as a mean line of the torus enveloping these orbits. If ω_0 is small enough, there are orbits, which sometimes leave this torus. However, for larger central frequencies, the spiral representing the Morlet wavelet on a phase plane fills almost a disk [see Fig. 2(b)]. Thus, it has some common tangents with all possible orbits. The uncertainty range corresponding to this threshold ω_0 determines the minimal size of the minimal bounding torus for an analyzed dynamical system. Any further growth of a central frequency does not lead to enveloping new orbits but simply increase the number of tangent points.

In the other words, below this threshold the continuous wavelet transform with the Morlet wavelet detects pulses of a width comparable to the oscillation period, i.e., individual unstable orbits, and, since the signals are chaotic, the localization of pulses in $x_{1,2}(t)$ fluctuates, and the phase difference accumulates accordingly. With increasing ω_0, time resolution degenerates and the wavelet length spans an increasing number of cycles. Consequently, phase difference fluctuations are increasingly smoothed by diffusion averaging, i.e., the consideration of individual orbits is replaced by tracing of tori including a bundle of them.

As a result, time-scale synchronization is detected. Nevertheless, it is not a pure chaotic synchronization in the classical sense [1] but a phase synchronization of averaged motions connected with another object, bounding tori, belonging to a coarser topological level. Thus, the wavelet-based method with a variable central frequency can be used to determine the minimum smoothing window size required to construct the bounding tori for a set of orbits from data.

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[1] A. Pikovsky, M. Rosenblum, and J. Kurths, *Synchronization: A Universal Concept in Nonlinear Sciences* (Cambridge University Press, New York, 2001).
[2] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).
[3] T. Kreuz, F. Mormann, R. G. Andrzejak, A. Kraskov, K. Lehnertz, and P. Grassberger, Physica D **225**, 29 (2007).
[4] M. Le Van Quyen, J. Foucher, J.-P. Lachaux, E. Rodriguez, A. Lutz, J. Martinerie, and F. J. Varela, J. Neurosci. Methods **111**, 83 (2001).
[5] M. G. Rosenblum, A. S. Pikovsky, J. Kurths, G. V. Osipov, I. Z. Kiss, and J. L. Hudson, Phys. Rev. Lett. **89**, 264102 (2002).
[6] A. A. Koronovskii and A. E. Hramov, JETP Lett. **79**, 316 (2004).
[7] A. E. Hramov and A. A. Koronovskii, Physica D **206**, 252 (2005).
[8] A. E. Hramov, A. A. Koronovskii, V. I. Ponomarenko, and M. D. Prokhorov, Phys. Rev. E **75**, 056207 (2007).
[9] D. Ghosh, A. Ray, and A. R. Chowdhury, Mod. Phys. Lett. B **22**, 1867 (2008).
[10] E. B. Postnikov, J. Exp. Theor. Phys. **105**, 652 (2007).
[11] S. Mallat, *A Wavelet Tour of Signal Processing* (Academic Press, New York, 1999).
[12] I. De Moortel, S. A. Munday, and A. W. Hood, Sol. Phys. **222**, 203 (2004).
[13] P. S. Addison, J. N. Watson, and T. Feng, J. Sound Vib. **254**, 733 (2002).
[14] J. J. Koenderink, Biol. Cybern. **50**, 363 (1984).
[15] T. D. Tsankov and R. Gilmore, Phys. Rev. Lett. **91**, 134104 (2003).
[16] C. Letellier, E. Roulin, and O. E. Rössler, Chaos, Solitons Fractals **28**, 337 (2006).
[17] H. Giacomini and S. Neukirch, Phys. Lett. A **227**, 309 (1997).